## 1.3 Problems NS-3

**Topic of this homework:** Pythagorean triplets, Pell's equation, Fibonacci sequence

### Pythagorean triplets

**Problem #** *1: Euclid's formula for the Pythagorean triplets* $a, b, c$ *is* $a = p^2 - q^2$, $b = 2pq$, *and* $c = p^2 + q^2$.

*– 1.1: What condition(s) must hold for* $p$ *and* $q$ *such that* $a$, $b$, *and* $c$ *are always positive and nonzero?*

**Sol:** $p > q > 0$ (strictly greater than)

*– 1.2: Solve for* $p$ *and* $q$ *in terms of* $a$, $b$, *and* $c$.

**Sol:**

**Method 1:** Given $a, c$, one may find $p, q$ via matrix operations by solving the *nonlinear system of equations* for $p, q$.

First solve linear system of equations for $p^2, q^2$:

$$\begin{bmatrix} a \\ c \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} p^2 \\ q^2 \end{bmatrix}$$

Inverting this 2x2 matrix gives (the determinant $\Delta = 2$)

$$\begin{bmatrix} p^2 \\ q^2 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} a \\ c \end{bmatrix}.$$

Thus $p = \pm\sqrt{(a+c)/2}$, $q = \pm\sqrt{(c-a)/2}$.

**Method 2:** The algebraic approach is:

$$a + c = (p^2 - q^2) + (p^2 + q^2) = 2p^2$$

$$-a + c = -(p^2 - q^2) + (p^2 + q^2) = 2q^2,$$

Thus $p = \sqrt{(a+c)/2}$, $q = \sqrt{(c-a)/2}$, where $p, q \in \mathbb{N}$.

Method 1 seems more "transparent" than Method 2.

**Problem #** *2: The ancient Babylonians (ca. 2000 BCE) cryptically recorded $(a, c)$ pairs of numbers on a clay tablet, archeologically denoted* Plimpton-322 *(see* **??***).*

– *2.1: Find $p$ and $q$ for the first five pairs of $a$ and $c$ shown here from Plimpton-322.*

| $a$ | $c$ |
|---|---|
| 119 | 169 |
| 3367 | 4825 |
| 4601 | 6649 |
| 12709 | 18541 |
| 65 | 97 |

Find a formula for $a$ in terms of $p$ and $q$.
**Sol:**

$$(a, c) = (119, 169) \qquad (p, q) = \pm(12, 5)$$
$$(a, c) = (3367, 4825) \qquad (p, q) = \pm(64, 27)$$
$$(a, c) = (4601, 6649) \qquad (p, q) = \pm(75, 32)$$
$$(a, c) = (12709, 18541) \qquad (p, q) = \pm(125, 54)$$
$$(a, c) = (65, 97) \qquad (p, q) = \pm(9, 4)$$

– *2.2: Based on Euclid's formula, show that $c > (a, b)$.*

**Sol:** $c - a = (p^2 + q^2) - (p^2 - q^2) = 2q^2$
Because $2q^2$ is always positive, $c > a$
$c - b = (p^2 + q^2) - 2pq = (p - q)^2 > 0$
Note that by the definition of $p, q \in \mathbb{N}$, $p > q$.

– *2.3: What happens when $c = a$?*

**Sol:** Then its not a triangle since $b = 0$. The triangle is degenerate.

– *2.4: Is $b + c$ a perfect square? Discuss.*

**Sol:** $b + c = p^2 + 2pq + q^2 = (p + q)^2$. Since $p$ and $q$ are integers, $b + c$ will always be a perfect square ($\sqrt{b + c}$ will always be an integer).

## Pell's equation:

**Problem #** *3: Pell's equation is one of the most historic (i.e., important) equations of Greek number theory because it was used to show that $\sqrt{2} \in \mathbb{I}$. We seek integer solutions of*

$$x^2 - Ny^2 = 1.$$

As shown in Sec. **??**, the solutions $x_n, y_n$ for the case of $N = 2$ are given by the linear $2 \times 2$ matrix recursion

$$\begin{bmatrix} x_{n+1} \\ y_{n+1} \end{bmatrix} = 1\jmath \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x_n \\ y_n \end{bmatrix}$$

with $[x_0, y_0]^T = [1, 0]^T$ and $1\jmath = \sqrt{-1} = e^{\jmath\pi/2}$. It follows that the general solution to Pell's equation for $N = 2$ is

$$\begin{bmatrix} x_n \\ y_n \end{bmatrix} = (e^{\jmath\pi/2})^n \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}^n \begin{bmatrix} x_0 \\ y_0 \end{bmatrix}.$$

To calculate solutions to Pell's equation using the matrix equation above, we must calculate

$$A^n = e^{\jmath\pi n/2} \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}^n = e^{\jmath\pi n/2} \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix} \cdots \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix},$$

which becomes tedious for $n > 2$.

*– 3.1: Find the companion matrix and thus the matrix $A$ that has the same eigenvalues as Pell's equation.* Hint: Use Matlab's function `[E,Lambda] = eig(A)` to check your results!

**Sol:** The companion matrix is

$$A = \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}$$

*– 3.2: Solutions to Pell's equation were used by the Pythagoreans to explore the value of $\sqrt{2}$. Explain why Pell's equation is relevant to $\sqrt{2}$.*

**Sol:** As discussed in Sec. 2.5.2, as the iteration $n$ increases, the ratio of the $x_n/y_n$ approaches $\sqrt{2}$.

*– 3.3: Find the first three values of $(x_n, y_n)^T$ by hand and show that they satisfy Pell's equation for $N = 2$.* **Sol:** *See class notes (slide 9.4.2) for this calculation. By hand, find the eigenvalues $\lambda_\pm$ of the $2 \times 2$ Pell's equation matrix*

$$A = \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}.$$

**Sol:** The eigenvalues are given by the roots of the equation $(1 - \lambda_\pm)^2 = 2$. Thus $\lambda_\pm = 1 \pm \sqrt{2} = \{2.1412, -.4142\}$

*– 3.4: By hand, show that the matrix of eigenvectors, $E$, is*

$$E = \begin{bmatrix} \vec{e}_+ & \vec{e}_- \end{bmatrix} = \frac{1}{\sqrt{3}} \begin{bmatrix} -\sqrt{2} & \sqrt{2} \\ 1 & 1 \end{bmatrix}.$$

**Sol:**

The eigenvectors $\vec{e}_\pm$ may be found by solving

$$A \begin{bmatrix} e_1 \\ e_2 \end{bmatrix} = \lambda_\pm \begin{bmatrix} e_1 \\ e_2 \end{bmatrix} \to (A - \lambda_\pm I) \begin{bmatrix} e_1 \\ e_2 \end{bmatrix} = 0$$

For $\lambda_+$, this gives

$$0 = \begin{bmatrix} 1-(1+\sqrt{2}) & 2 \\ 1 & 1-(1+\sqrt{2}) \end{bmatrix} \begin{bmatrix} e_1 \\ e_2 \end{bmatrix} = \begin{bmatrix} -\sqrt{2} & 2 \\ 1 & -\sqrt{2} \end{bmatrix} \begin{bmatrix} e_1 \\ e_2 \end{bmatrix}$$

which gives the relation between the elements of $\vec{e}_+$, $e_1, e_2$, as $e_1 = \sqrt{2}e_2$.
The eigenvectors are defined to be unit length and orthogonal, namely

1. $||\vec{e}_k||^2 = \vec{e}_k \cdot \vec{e}_k = 1$
2. $\vec{e}_+ \cdot \vec{e}_- = 0$.

Once we normalize $\vec{e}_+$ to have unit length, we obtain the first eigenvector

$$\vec{e}_+ = \frac{1}{\sqrt{3}} \begin{bmatrix} -\sqrt{2} \\ 1 \end{bmatrix}$$

Repeating this for $\lambda_-$ gives

$$\vec{e}_- = \frac{1}{\sqrt{3}} \begin{bmatrix} \sqrt{2} \\ 1 \end{bmatrix}$$

Thus, the matrix of eigenvalues is

$$E = \frac{1}{\sqrt{3}} \begin{bmatrix} -\sqrt{2} & \sqrt{2} \\ 1 & 1 \end{bmatrix}$$

*– 3.5: Using the eigenvalues and eigenvectors you found for $A$, verify that*

$$E^{-1}AE = \Lambda \equiv \begin{bmatrix} \lambda_+ & 0 \\ 0 & \lambda_- \end{bmatrix}$$

**Sol:** Using the formula for a matrix inverse, we find

$$E^{-1} = \frac{1}{\det(E)}\begin{bmatrix} e_{22} & -e_{12} \\ -e_{21} & e_{11} \end{bmatrix} = \frac{3}{-2\sqrt{2}}\frac{1}{\sqrt{3}}\begin{bmatrix} 1 & -\sqrt{2} \\ -1 & -\sqrt{2} \end{bmatrix} = \frac{-\sqrt{3}}{2\sqrt{2}}\begin{bmatrix} 1 & -\sqrt{2} \\ -1 & -\sqrt{2} \end{bmatrix}$$

Thus

$$\begin{aligned} E^{-1}AE &= \frac{-\sqrt{3}}{2\sqrt{2}}\begin{bmatrix} 1 & -\sqrt{2} \\ -1 & -\sqrt{2} \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}\frac{1}{\sqrt{3}}\begin{bmatrix} -\sqrt{2} & \sqrt{2} \\ 1 & 1 \end{bmatrix} \\ &= \frac{-1}{2\sqrt{2}}\begin{bmatrix} 1 & -\sqrt{2} \\ -1 & -\sqrt{2} \end{bmatrix}\begin{bmatrix} (-\sqrt{2}+2) & (\sqrt{2}+2) \\ (-\sqrt{2}+1) & (\sqrt{2}+1) \end{bmatrix} \\ &= \begin{bmatrix} 1-\sqrt{2} & 0 \\ 0 & 1+\sqrt{2} \end{bmatrix} = \Lambda \end{aligned}$$

*– 3.6: Once you have diagonalized $A$, use your results for $E$ and $\Lambda$ to solve for the $n = 10$ solution $(x_{10}, y_{10})^T$ to Pell's equation with $N = 2$.*
**Sol:** $x_{10} = -3363$ and $y_{10} = -2378$. Note this formulation gives the negative solution, but since the values for $n = 10$ are real, when they are squared in Pell's equation, it makes no difference whether they are negative or positive.

## The Fibonacci sequence

The Fibonacci sequence is famous in mathematics and has been observed to play a role in the mathematics of genetics. Let $x_n$ represent the Fibonacci sequence,

$$x_{n+1} = x_n + x_{n-1}, \tag{NS-3.1}$$

where the current input sample $x_n$ is equal to the sum of the previous two inputs. This is a "discrete time" recurrence relationship. To solve for $x_n$, we require some initial conditions. In this exercise, let us define $x_0 = 1$ and $x_{n<0} = 0$. This leads to the Fibonacci sequence $\{1, 1, 2, 3, 5, 8, 13, \ldots\}$ for $n = 0, 1, 2, 3, \ldots$.

Equation NS-3.1 is equivalent to the $2 \times 2$ matrix equations

$$\begin{bmatrix} x_n \\ y_n \end{bmatrix} = A\begin{bmatrix} x_{n-1} \\ y_{n-1} \end{bmatrix}, \qquad A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}. \tag{NS-3.2}$$

**Problem #** *4: Here we seek the general formula for $x_n$. Like Pell's equation, the Fibonacci equation has a recursive eigenanalysis solution. To find it we must recast $x_n$ as a $2 \times 2$ matrix relationship and then proceed, as we did for the Pell case.*

*– 4.1: Show that the Fibonacci sequence $x_n = x_{n-1} + x_{n-2}$ may be generated by*

$$\begin{bmatrix} x_n \\ y_n \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}^n \begin{bmatrix} x_0 \\ y_0 \end{bmatrix}, \qquad \begin{bmatrix} x_0 \\ y_0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}. \tag{NS-3.3}$$

**Sol:** Using the Matrix Eigen-equation, powers of the eigen equation $\mathbf{A}^n = \mathbf{E}\Lambda^n\mathbf{E}^{-1}$. The final solution is

$$\begin{bmatrix} x_n \\ y_n \end{bmatrix} = \boldsymbol{E}\begin{bmatrix} \lambda_+ & 0 \\ 0 & \lambda_- \end{bmatrix}^n \boldsymbol{E}^{-1}\begin{bmatrix} x_0 \\ y_0 \end{bmatrix}. \tag{NS-3.4}$$

*– 4.2: What is the relationship between $y_n$ and $x_n$?*
**Sol:** This equation says that $x_n = x_{n-1} + y_{n-1}$ and $y_n = x_{n-1}$. The latter equation may be rewritten as $y_{n-1} = x_{n-2}$. Thus

$$x_n = x_{n-1} + x_{n-2}$$

as requested.

*– 4.3: Write a Matlab/Octave program to compute $x_n$ using the matrix equation above. Test your code using the first few values of the sequence. Using your program, what is $x_{40}$?* Note: Consider using the eigenanalysis of $A$, described by Eq. **??** of the text.

**Sol:** You can try something like:

```
function xn = fib(n)
A = [1 1; 1 0]; [E,D] = eig(A); xy = E*D^n*inv(E)*[1; 0];
xn = xy(1);
```

For this initial condition, $x_{40} = 165,580,141 = \frac{1}{\sqrt{5}}\left(\frac{(1+\sqrt(5)}{2}\right)^{41}$.

*– 4.4: Using the eigenanalysis of the matrix $A$ (and a lot of algebra), show that it is possible to obtain the general formula for the Fibonacci sequence*

$$x_n = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^{n+1} - \left(\frac{1-\sqrt{5}}{2}\right)^{n+1}\right]. \qquad \text{(NS-3.5)}$$

*– 4.5: What are the eigenvalues $\lambda_\pm$ of the matrix $A$?*

**Sol:** The eigenvalues of the Fibonacci matrix are given by

$$det\begin{bmatrix}1-\lambda & 1\\ 1 & -\lambda\end{bmatrix} = \lambda^2 - \lambda - 1 = (\lambda - 1/2)^2 - (1/2)^2 - 1 = (\lambda - 1/2)^2 - 5/4 = 0,$$

thus $\lambda_\pm = \frac{1\pm\sqrt{5}}{2} = [1.618, -0.618]$.

*– 4.6: How is the formula for $x_n$ related to these eigenvalues?* Hint: Find the eigenvectors.

**Sol:** The eigenvectors (determined from the equation $(A - \lambda_\pm I)\vec{e}_\pm = \vec{0}$, and normalized to 1) are given by

$$\vec{e}_+ = \begin{bmatrix}\frac{\lambda_+}{\sqrt{\lambda_+^2+1}}\\ \frac{1}{\sqrt{\lambda_+^2+1}}\end{bmatrix} \qquad \vec{e}_- = \begin{bmatrix}\frac{\lambda_-}{\sqrt{\lambda_-^2+1}}\\ \frac{1}{\sqrt{\lambda_-^2+1}}\end{bmatrix} \qquad E = \begin{bmatrix}\vec{e}_+ & \vec{e}_-\end{bmatrix}$$

From the eigenanalysis, we find that

$$\begin{bmatrix}x_n\\ y_n\end{bmatrix} = E\begin{bmatrix}\lambda_+^n & 0\\ 0 & \lambda_-^n\end{bmatrix}E^{-1}\begin{bmatrix}1\\0\end{bmatrix} = \begin{bmatrix}e_{11} & e_{12}\\ e_{21} & e_{22}\end{bmatrix}\begin{bmatrix}\lambda_+^n & 0\\ 0 & \lambda_-^n\end{bmatrix}\frac{1}{(e_{11}e_{22} - e_{12}e_{21})}\begin{bmatrix}e_{22} & -e_{12}\\ -e_{21} & e_{11}\end{bmatrix}\begin{bmatrix}1\\0\end{bmatrix}.$$

Solving for $x_n$ we find that

$$\begin{aligned} x_n &= \frac{1}{(e_{11}e_{22} - e_{12}e_{21})}\left(\lambda_+^n e_{11}e22 - \lambda_-^n e_{12}e_{21}\right)\\ &= \frac{1}{\frac{\sqrt{5}}{\sqrt{(\lambda_+^2+1)(\lambda_-^n+1)}}}\left[\lambda_+^n\left(\frac{\lambda_+^n}{\sqrt{(\lambda_+^2+1)(\lambda_-^n+1)}}\right) - \lambda_-^n\left(\frac{\lambda_-^n}{\sqrt{(\lambda_+^2+1)(\lambda_-^n+1)}}\right)\right]\\ &= \frac{1}{\sqrt{5}}\left[\lambda_+^{n+1} - \lambda_-^{n+1}\right]\end{aligned}$$

*– 4.7: What happens to each of the two terms*

$$\left[\left(1 \pm \sqrt{5}\right)/2\right]^{n+1}?$$

**Sol:** $[(1+\sqrt{5})/2]^{n+1} \to 0$ and $[(1+\sqrt{5})/2]^{n+1} \to \infty$

*– 4.8: What happens to the ratio $x_{n+1}/x_n$?*

**Sol:** $x_{n+1}/x_n \to (1+\sqrt{5})/2$, because $\left((1-\sqrt{5})/2\right)^n \to 0$ as $n \to \infty$ thus for large $n$, $x_n \approx [(1+\sqrt{5})/2]^{n+1}$.

**Problem #** *5: Replace the Fibonacci sequence with*

$$x_n = \frac{x_{n-1} + x_{n-2}}{2},$$

such that the value $x_n$ is the average of the previous two values in the sequence.

*– 5.1: What matrix $A$ is used to calculate this sequence?*
**Sol:**

$$A = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ 1 & 0 \end{bmatrix}$$

*– 5.2: Modify your computer program to calculate the new sequence $x_n$. What happens as $n \to \infty$?*
**Sol:** As $n \to \infty$, $x_n \to 2/3$

*– 5.3: What are the eigenvalues of your new $A$? How do they relate to the behavior of $x_n$ as $n \to \infty$?* Hint: You can expect the closed-form expression for $x_n$ to be similar to Eq. NS-3.5.
**Sol:** The eigenvalues are $\lambda_+ = 1$ and $\lambda_- = -0.5$. From Eq. **??**, the expression for $A^n$ is

$$A^n = (E\Lambda E^{-1})^n = E\Lambda^n E^{-1} = \begin{bmatrix} \lambda_+ & 0 \\ 0 & \lambda_- \end{bmatrix}^n = \begin{bmatrix} \lambda_+^n & 0 \\ 0 & \lambda_-^n \end{bmatrix}.$$

The solution is the sum of two sequences, one a constant and the other an oscillation that quickly fades. As $n \to \infty$, $\lambda_+^n = 1^n \to 1$ and $\lambda_-^n = (-1/2)^n \to 0$. The solution becomes

$$x_n = \frac{2}{3}\left[\lambda_+^n - \lambda_-^n\right] = \frac{2}{3}\left[1^n - (-1)^n\right] \to \frac{2}{3}.$$

**Problem #** *6: Consider the expression*

$$\sum_1^N f_n^2 = f_N f_{N+1}.$$

*– 6.1: Find a formula for $f_n$ that satisfies this relationship. Hint: It holds for only the Fibonacci recursion formula.*
**Sol:** Write this out for $N$ and $N-1$:

$$\begin{aligned} f_1^2 + f_2^2 + \cdots + f_{N-1}^2 + f_N^2 &= f_N f_{N+1} \\ f_1^2 + f_2^2 + \cdots + f_{N-1}^2 \qquad &= f_{N-1} f_N \end{aligned}$$

Subtracting gives

$$\begin{aligned} & f_N^{\cancel{2}} = \cancel{f_N} f_{N+1} - f_{N-1}\cancel{f_N} = \cancel{f_N}(f_{N+1} - f_{N-1}) \\ & f_N = f_{N+1} - f_{N-1} \end{aligned}$$

Thus the relation only holds for the Fibonacci recursion formula.